ANALYSIS OF TRUNCATED FACTORIAL EXPERIMENTS

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INTRODUCTION

In factorial experiments demand on experimental resources increases with the number of factors due to large number of treatment combinations and the presence of considerable number of non-zero constituents in different treatment combinations. Finney (1945) suggested the concept of fractional replication for factorial experiments for reducing the first of the above two types of demands on resources. We have proposed truncated fractions of the factorial experiments for controlling both these factors. If there are m factors
2^m $A_1, A_2, \ldots, A_m$ at two levels each and n factors $B_1, B_2, \ldots, Bn$ at 3^n
three levels each then

$$a_1^{x_1}\, a_2^{x_2} \ldots . a_m^{x_m}\, b_1^{z_1}\, b_2^{z_2} \ldots . b_n^{z_n}$$

where

$$x_i = 0,1 \text{ and } z_j = 0,1,2,$$

denotes any treatment combination, then a k-letter truncated factorial experiment ($k-L.T.F.E$) will have all such treatment combinations for which $\Sigma x_i + \Sigma z_j \leqslant k$. If all b's are zeros then it will be k-letter truncated factorial experiment of 2^m factorial and on the other hand all a's are zeros the design will turn out to be a k-letter truncated factorial experiment of 3^n.

In the present paper a systematic procedure of obtaining the main effects and other interactions through such truncated experiments has been indicated for two-letter truncated factorial experiments of 2^m, 3^n and $3^n \times 2^m$ factorial experiments.

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2. Method of Estimation of the Main Effects and Two Factor Interaction Components.

(a) 2^m Two-letter truncated design

Let there be m factors $A_1, A_2, \ldots, A_m$ each at two levels.

Let

$a_1^{x_1} a_2^{x_2} \ldots a_m^{x_m}$ denote the treatment in which the factors

$A_1, A_2, \ldots, A_m$ occur at levels

$x_1, x_2, \ldots, x_m$ $(x_i = 0, 1)$

when

$a_i^{\circ} = 1$

and

$1.a_i^{x_i} = a_i^{x_i}$

In accordance with the standard convention we shall represent a mean response to a treatment by the same symbol of the treatment.

Let any interaction be denoted by $A_1^{\lambda_1} A_2^{\lambda_2} \ldots A_m^{\lambda_m}$ $(\lambda_i = 0, 1)$

where $A_i^0 = 1$, $1.A_i^{\lambda_i} = A_i^{\lambda_i}$

Taking the usual line as factorial fixed model along with the restrictions on interaction effects we have

$$E\left\{\begin{pmatrix} 1 \\ a_i \end{pmatrix} \oplus \begin{pmatrix} 1 \\ a_2 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} 1 \\ a_m \end{pmatrix}\right\}$$

$$= H^m\left\{\begin{pmatrix} 1 \\ A_1 \end{pmatrix} \oplus \begin{pmatrix} 1 \\ A_2 \end{pmatrix} \oplus \begin{pmatrix} 1 \\ A_n \end{pmatrix}\right\}$$

where

$$H = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} Co\begin{pmatrix} \\ 1 \end{pmatrix} & C1\begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ Co & C1 \end{bmatrix} \text{say}$$

we can write any treatment yield

$$a_1^{x_1} a_2^{x_2} \ldots a_m^{x_m} = \prod_{i=1}^{m} \{Co(x_i) + C_1(x_i)A_i\} \quad \ldots (1)$$

Thus the coefficient of any interaction

$A_1^{\lambda_1} A_2^{\lambda_2} \ldots A_m^{\lambda_m}$ on the right hand side is

$C_{\lambda_1}(x_1)C_{\lambda_2}(x_2) \ldots C_{\lambda_m}(x_m)$

Let A be a column vector whose elements are the interactions in some order and A' the corresponding row vector.

Now

if $X'=(x_1, x_2 \ldots x_m)$

We can write (1) in a compact form

$$a^{x1}=h\,(X')\,A,$$

where

$h(X')$ is the row vector of the coefficients in (1).

Following this procedure of writing treatment yields as a function of the interactions and expressing the yield responses (*i.e.* treatment yield—control yield) instead of actual yield for the two-letter truncated factorial experiments and assuming that interactions of order two or more are negligible, we get the following form of the $h\,(X')$ for the two-letter truncated factorial experiment for 2^m.

$$L(x')=\left[\begin{array}{c|c} I_{11} & -M_{21}^T \\ \hline M_{21} & N_{22} \end{array}\right]$$

where I_{11} stands for unit matrix of order $m_{c1} \times m_{c1}$

M_{21} is matrix of order $m_{c2} \times m_{c1}$ where m_{c2} rows are obtained by adding the rows of the unit matrix in a natural order.

N_{22} is a matrix of the order $m_{c2} \times m_{c2}$ obtained from $-M_{21}^T$ in the same way as M_{21} from I_{11}

Since $a^{x'}=h(X')\,A$

$$A=[h(X')]^{-1}(a^{x'})$$

It can be easily verified that

$$[h(X')]^{-1}=K\left[\begin{array}{c|c} \mu_{11} & M_{12} \\ \hline -M_{12}^T & I_{22} \end{array}\right]$$

where any element $\mu ij=3-n$ for $i=j$

$=-1$ for $i \neq j$

and other submatrices are already defined.

Since each interaction is a contrast in terms of original observations we may express the variance of any treatment of the row corresponding to $A_i^{\lambda_1} \; A_j^{\lambda_2}$

by $\quad V(A_i^{\lambda_1} \; A_j^{\lambda_2}) = [\Sigma 1_i^2 + (\Sigma 1_i)^2]\sigma^2$

where $\quad 1_i$ represents elements in the row

and $\quad \sigma^2$ is the per plot variance.

(*b*) 3^n *Two-letter truncated Factorial Experiment.*

Suppose there are n factors $B_1, B_2, \ldots, B_n$ each at three levels.

Let $\quad b_1^{z_1} \; b_2^{z_2} \ldots b_n^{z_n}$ $(z_j = 0, 1, 2)$ denote the treatment in which the factors $B_1, B_2, \ldots, B_n$ occur at levels $z_1, z_2, \ldots, z_n$ $(z_j = 0, 1, 2)$

where $\quad b_j^0 = 1$

and $\quad 1.\, b_j^{z_j} = b_j^{z_j}$

for $\quad z_j \neq 0.$

Thus if as usual we represent a treatment or the mean response by the symbol

$$Y(Z_1, Z_2, \ldots, Z_n)$$

then, $E[Y(Z_1, Z_2, \ldots, Z_n)] = b_1^{z_1} \; b_2^{z_2} \ldots b_n^{z_n}$

Let any interaction be denoted by $B_1^{\mu_1} \; B_2^{\mu_2} \ldots B_n^{\mu_n}$ $(\mu_j = 0, 1, 2)$.

It can clearly be seen that the normal equations are of the form:

$$E\left[\begin{pmatrix} 1 \\ b_1 \\ b_2^2 \end{pmatrix} \oplus \begin{pmatrix} 1 \\ b_2 \\ b_2^2 \end{pmatrix} \oplus \quad \oplus \begin{pmatrix} 1 \\ b_n \\ b_n^2 \end{pmatrix}\right] = K^n \left\{\begin{pmatrix} 1 \\ B_1 \\ B_1^2 \end{pmatrix} \oplus \begin{pmatrix} 1 \\ B_2 \\ B_2 \end{pmatrix} \oplus \ldots \oplus \begin{pmatrix} 1 \\ B_n \\ B_n^2 \end{pmatrix}\right\}$$

with the convention $I.I = I$, $I.B = B.I$, $B^0 = I$

where

$$K = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & -2 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} d_0 \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} d_1 \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} d_2 \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \end{bmatrix} \text{ say}$$

Thus as before I is mean response, and effects defined here do not obey the convention that the sum of the positive coefficient is unity. Each effect represents a single degree of freedom. Thus

B_1B_2 is the interaction of linear component of B_1 and the linear component of B_2 and $B_1B_2^2$ the linear component of B_1 and the quadratic component of B_2.

Thus it can be deduced that

$$b_1^{z_1}\ b_2^{z_2} \ldots\ldots b_n^{z_n} = \prod_{j=1}^{n} \left\{ do\ (Z_i) + d_1\ (Z_j) + d_2(Z_j)B_j^2 \right\} \ldots (2)$$

The coefficient of $B_1^{\mu_1}\ B_2^{\mu_2} \ldots\ldots B_n^{\mu_n}$

when the right hand side of the above expression is expanded.

$d\mu_1\ (Z_1)\ d\mu_2\ (Z_2) \ldots\ldots d\mu_n\ (Z_n)$

As before we fix a standard order in interaction as $I, B_1, B_2 \ldots,$ $B_n, B_1^2, B_2^2 \ldots\ldots B_n^2, B_1B_2, B_1B_3 \ldots\ldots, B_{n-1}B_n, \ldots\ldots$ in which we want to take the interaction. Let B be the column vector of interaction in the same order and $Z'=(Z_1, Z_2 \ldots\ldots Z_n)$ then we can write the equation (2) in a compact form $b^{z'}=K(Z')B$

Now when we consider two-letter truncated factorial experiment for 3^n series assuming the interaction $B_1^{\mu_1}\ B_2^{\mu_2} \ldots\ldots B_n^{\mu_n}$ such that $\Sigma\mu_j \leqslant 2$ as negligible we get the following form of the $K(Z')$ split into 9 submatrices.

$$K(Z')=\left[\begin{array}{c|c|c} I_{11} & -3I_{11} & -M_{21}{}^T \\ \hline 2I_{11} & 0_{11} & -2M_{21}{}^T \\ \hline M_{21} & -2M_{21} & N_{22} \end{array}\right]$$

where I and 0 are the unit and zero matrices of the order indicated. M_{21} is the same matrix as defined for 2^m two-letter truncated factorial experiment, the N_{22} is a matrix in which any element

$N_{ij}=0$ (if union between row number and the column number is empty)

$=-1$ otherwise.

Now to estimate various effects we have to find the $K(Z')^{-1}$ because $B=[K(Z')]^{-1}\ b'$. The inverse of $K\ (Z')$ can be easily obtained as

$$K(Z')=\left[\begin{array}{c|c|c} -\mu_{11} & 1/2\ I_{11} & M_{21}{}^T \\ \hline -1/3 I_{11} & 1/6\ I_{11} & 0_{21}{}^T \\ \hline -M_{21} & 0_{21} & I_{22} \end{array}\right]$$

where for μ_{11} any element $\mu_{ij} = (n-1)$ when $i=j$ $(n>2)$

$$=1 \text{ when } i \neq j$$

For $n=2$ however there is an exception *i.e.* in this case

$$K(Z')=\left[\begin{array}{cc|cc|c} 1 & 0 & -3 & 0 & -1 \\ 0 & 1 & 0 & -3 & -1 \\ \hline 2 & 0 & 0 & 0 & -2 \\ 0 & 2 & 0 & 0 & -2 \\ \hline 1 & 1 & -3 & -3 & 0 \end{array}\right]$$

and

$$[K(Z')]^{-1}=1/6\left[\begin{array}{cc|cc|c} -3 & -3 & 3 & 0 & 3 \\ -3 & -3 & 0 & 3 & 3 \\ \hline -2 & 0 & 1 & 0 & 0 \\ 0 & -2 & 0 & 1 & 0 \\ \hline -3 & -3 & 0 & 0 & 3 \end{array}\right]$$

(c) *$2^m \times 3^n$ two-letter truncated factorial experiments.*

Let there be m factors $A_1, A_2 \ldots, A_m$ each at two levels and n factors $B_1, B_2, \ldots, B_n$ each at three levels ; then there are $2^m \times 3^n$ treatment combinations in a complete factorial. The symbol $a_1^{x_1}\ a_2^{x_2} \ldots am^{x_m}\ b_1^{z_1} b_2^{z_2} \ldots b_n^{z_n} \ldots$(3) denotes the treatment in which factors $A_1, A_2, \ldots, A_m$ occur at levels $x_1, x_2, \ldots, x_m (x_i=0,1)$ and the factors $B_1, B_2, \ldots, B_n$ occur at levels $Z_1, Z_2, \ldots Z_n$ $(Z_j=0,1,2)$.

In this case the relation between the treatment and interaction is :

$$E\left[(a^1{}_1)\oplus(a^1{}_2)\oplus\ldots\oplus(a^1{}_m)\oplus\begin{pmatrix}1\\ b_1\\ b^2{}_1\end{pmatrix}\oplus\begin{pmatrix}1\\ b_2\\ b^2{}_2\end{pmatrix}\oplus\ldots\oplus\begin{pmatrix}1\\ b_n\\ b^2{}_n\end{pmatrix}\right]=$$

$$G\left[(A^1{}_1)\oplus(A^1{}_2)\oplus\ldots\oplus(A^1{}_m)\oplus\begin{pmatrix}1\\ B^1\\ 6^2{}_1\end{pmatrix}\oplus\begin{pmatrix}1\\ B_2\\ B^2{}_2\end{pmatrix}\oplus\ldots\oplus\begin{pmatrix}1\\ B_n\\ B^2{}_n\end{pmatrix}\right]$$

with convention *I.I=I etc.* $A_1 B_1$=stands for $A_1 \times B_1$ linear ; similarly $B_1\ B_2$ stands for $B_1 \times B_2$ linear. From this we may express any treatment as linear combinations of interactions.

$$a_1^{x_1}\ a_2^{x_2} \ldots a_m^{x_m}\, b_1^{z_1}\ b_2^{z_2} \ \ldots\ b_n^{z_n} = \prod_{i=1}^{m} \left[C_i(x_i) + C_1(x_i) A_i \right]$$

$$\times \prod_{j=1}^{n} \left[do(z_j) + d_1(z_j) B_j + d_2(z_j) B_j^2 \right]$$

If we take some standard order of the interactions $A_1^{\lambda_1}\ A_2^{\lambda_2} \ldots A_m^{\lambda_m}\, B_1^{\mu_1}\ B_2^{\mu_2} \ldots B_n^{\mu_n}$ and let AB may be a column vector of that order and let the corresponding standard order for the treatments be ab in the form column vectors.

$$(ab') = (G)\ (AB')$$

Now for the two-letter truncated factorial experiments in 2^m x 3^n series (see page 43)
where

(*i*) $M_{(n_2)(n_1)}$ is the matrix of order $(n_2) \times (n_1)$ having the elements in the fashion given for 2^n fact experiment.

(*ii*) $\alpha_1 =$ n-vector (1,1 . . . . 1)

(*iii*) $\alpha_o =$ n-nector (0,0 . . . . 0)

(*iv*) $I\ (\alpha_1, \alpha_0)$ is a unit matrix of order $m \times m$ with diagonal elements as vector and the remaining elements as vector .

(*v*) $P\binom{m}{2}\binom{m}{2}$ is a matrix with any elements $pij = 0$ for $i=j$

$= -2$ for $i \neq j$

(*vi*) is a systematic matrix with diagonal elements as matrix and other elements as $(-i)$ I matrix of $T\, m \times m$ order $n \times n$.

Since $(ab\ ) = (G)\ (AB')$

$$(AB') = (G)^{-1}\ (ab')$$

We can easily get the inverted matrix as follows (see page 44):

where

$\lambda\binom{m}{1}\binom{n}{1}$ is a matrix with the elements

$\lambda_{ii} = [\ 1 - (n+m)]/4$ if n is even

$= -(m+n)/4$ if n is odd

$$
[G]^{-1}=\left[\begin{array}{c|c|c|c|c|c}
Q_{\binom{n}{1}\binom{n}{1}} & 1/2I_{\binom{n}{1}\binom{n}{1}} & M^T_{\binom{n}{2}\binom{n}{1}} & -1/2K^T_{\binom{n}{1}\binom{n}{1}} & O^T_{\binom{m}{2}\binom{n}{1}} & 1/2I_{\binom{n}{1}\binom{n}{1}} \\ \hline
-1/3I_{\binom{n}{1}\binom{n}{1}} & 1/6I_{\binom{n}{1}\binom{n}{1}} & O^T_{\binom{n}{2}\binom{n}{1}} & O^T_{\binom{n}{1}\binom{n}{1}} & O^T_{\binom{m}{2}\binom{n}{1}} & O_{\binom{n}{1}\binom{n}{1}} \\ \hline
-M_{\binom{n}{2}\binom{n}{1}} & O_{\binom{n}{2}\binom{n}{1}} & I_{\binom{n}{2}\binom{n}{2}} & O^T_{\binom{m}{2}\binom{n}{1}} & O^T_{\binom{m}{2}\binom{n}{2}} & O_{\binom{n}{2}\binom{n}{1}} \\ \hline
-1/2K_{\binom{m}{1}\binom{n}{1}} & O_{\binom{m}{1}\binom{n}{1}} & O_{\binom{m}{2}\binom{n}{1}} & \lambda_{\binom{m}{1}\binom{n}{1}} & -1/4M^T_{\binom{m}{1}\binom{m}{1}} & -1/2I_{(\alpha_1,\ \alpha_0)} \\ \hline
O_{\binom{m}{2}\binom{n}{1}} & O_{\binom{m}{2}\binom{n}{1}} & O_{\binom{m}{2}\binom{n}{2}} & -1/4M_{\binom{m}{2}\binom{m}{1}} & 1/4I_{\binom{m}{2}\binom{n}{2}} & O^T_{\binom{n}{1}\binom{m}{2}} \\ \hline
-1/2I_{\binom{n}{1}\binom{n}{1}} & O_{\binom{m}{1}\binom{n}{1}} & O_{\binom{n}{1}\binom{n}{2}} & -1/2I_{(\alpha_1,\ \alpha_0)\binom{n}{1}\binom{m}{1}} & O_{\binom{n}{1}\binom{m}{2}} & 1/2I_{m\times n}
\end{array}\right]
$$

Repeated m times.

Repeated m times

$$
G = \left[\begin{array}{ccc|c|c|c}
I_{\binom{n}{1}\binom{n}{1}} & -3I_{\binom{n}{1}\binom{n}{1}} & -M^T_{\binom{n}{2}\binom{n}{1}} & & & -I_{\binom{n}{1}\binom{n}{1}} \\
\hline
2I_{\binom{n}{1}\binom{n}{1}} & O_{\binom{n}{1}\binom{n}{1}} & -2M^T_{\binom{n}{2}\binom{n}{1}} & O^T_{\binom{m}{1}\frac{n(n+3)}{2}} & O^T_{\binom{m}{2}\frac{n(n+3)}{2}} & -2I_{\binom{n}{1}\binom{n}{1}} \\
\hline
M_{\binom{n}{2}\binom{n}{1}} & -3M_{\binom{n}{2}\binom{n}{1}} & N_{\binom{n}{2}\binom{n}{2}} & & & -M_{\binom{n}{2}\binom{n}{1}} \\
\hline
\multicolumn{3}{c|}{O_{\binom{m}{1}\frac{n(n+3)}{2}}} & 2I_{\binom{m}{1}\binom{m}{1}} & -2M^T_{\binom{m}{2}\binom{m}{1}} & -2I^T_{(\alpha_1,\,\alpha_0)} \\
\hline
\multicolumn{3}{c|}{O_{\binom{m}{2}\frac{n(+3)}{2}}} & 2M_{\binom{m}{2}\binom{m}{1}} & P_{\binom{m}{2}\binom{m}{2}} & -2M^T_{(\alpha_1,\,\alpha_0)} \\
\hline
I_{\binom{n}{1}\binom{n}{1}} & -3I_{\binom{n}{1}\binom{n}{1}} & -M^T_{\binom{n}{2}\binom{n}{1}} & -2I(\alpha_1,\,\alpha_0) & -2M_{(\alpha_1,\alpha_0)} & T_{m\times m}
\end{array}\right]
$$

Repeated m times (the first three block columns I, $-3I$, $-M^T$)

Repeated m times (the first three block rows of the last column $-I$, $-2I$, $-M$)

and

$$\lambda_{ij} = -1/4$$

$Q\binom{n}{1}\binom{n}{1}$ is a matrix with the elements

$$q_{ii} = -(m+n)/2 \text{ if } n \text{ is even}$$

$$= -(m+n+1)/2 \text{ if } n \text{ is odd}$$

and

$$q_{ij} = -1 \text{ if } (i \neq j)$$

3. Summary

In this paper we have defined *K*-letter truncated factorial experiments. The estimation procedure for main-effects and interactions (on Linear $\times$ linear component of two factor interactions if the factors are at more than two levels) has been given for 2—*L.T.F.E.* of 2^m, 3^n and 3^n *x* 2^m factorial experiments.

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